

Quantum filtering equations for system driven by non-classical fields

Anita Dąbrowska

Nicolaus Copernicus University, Toruń,
Collegium Medicum Bydgoszcz,
ul. Jagiellońska 15, 85-067 Bydgoszcz, Poland
(Dated:)

Using Gardiner and Collet's input-output model and the concept of cascade system, we determine the filtering equation for a system driven by non-classical states of light. We describe the stochastic evolution conditioned on the results of the quadrature measurement. The observed system and electromagnetic field are described by making use of quantum stochastic unitary evolution. We consider two examples of the non-classical states of the field: a combination of vacuum and single photon states and a mixture of two coherent states.

PACS numbers:

I. INTRODUCTION

Non-classical states of light such as continuous-mode Fock states [1] and their mixtures and superpositions arouse an interest because of their potential application in quantum metrology [2, 3] and quantum communication [4, 5]. Theoretical description of optimal excitation of quantum system by wave packet traveling in space became one of the essential aspects of the subject [6–10]. To examine excitation as well as process of storing information in a single ion, atom or quantum dot master equations description are usually considered. But instead of one equation we have here sets of coupled differential equations. Derivation of master equations for a quantum system interacting with a wave packet of light prepared in a single photon state one can find, for instance, in [12, 25]. And for a wave packet taken in a continuous-mode Fock states in [11].

A detailed investigation of the phenomena of excitation of quantum system by wave packet traveling in space requires consideration of individual quantum trajectories conditional on the results of measurement of the light emitted or scattered by the system. The main theme of this paper will be determination of stochastic differential equations for quantum system interacting with chosen non-classical states of light. We will use the theory of quantum state estimation based on the idea of the non-demolition observation [13] developed in the framework of quantum stochastic calculus [14, 15]. In this approach we deal with a quantum system interacting with an environment modeled by the Bose field [16–18, 23]. The Bose field, being an approximation of the electromagnetic field, disturbs the free evolution of the quantum system but at the same time it allows for an indirect and continuous in time observation of the system. Observation of the field after interaction with the system (the output field) provides us with the information about the system. Stochastic equation describing the time evolution of the quantum system conditioned on the results of the continuous in time measurement of the output field is called quantum filtering equation. Derivation of the

filtering equation for the Bose field in the vacuum state one can find, for instance, in [19, 20, 22, 23]. Determination of the conditional dynamics of the system interacting with the Bose field in the squeezed Gaussian state was given in [21], and for multi-channels case in [24]. Problem of derivation of filtering equations for non-Gaussian states such as continuous-mode single photon state or continuous-mode cat states one can find in [25, 26].

In this paper, we describe derivation of the filtering equation for two cases: for the case when the Bose field is taken in a combination of the vacuum and single photon states and for the case when the Bose field is in a mixture of two coherent cases. In order to specify the conditional evolution of the system we use the idea of introducing an ancilla being a source of the desired non-classical signals [12]. Finally we compare our results with the filter discussed in [25, 26].

II. FILTERING EQUATION FOR SYSTEM DRIVEN BY THE FIELD IN A COMBINATION OF VACUUM AND SINGLE PHOTON STATES

Let us consider a quantum system (we will call it \mathcal{S}) interacting with the Bose field. We assume that the evolution of the whole system (\mathcal{S} and the Bose field) is given by the unitary operator, U_t , satisfying the Itô quantum stochastic differential equation (QSDE) [14, 16]

$$\begin{aligned} dU_t &= \left[L dB_t^\dagger - L^\dagger S dB_t \right. \\ &\quad \left. + (S - I) d\Lambda_t - \left(iH + \frac{1}{2} L^\dagger L \right) dt \right] U_t, \\ U_{t=0} &= I, \end{aligned}$$

where L , S are operators acting in the Hilbert space \mathfrak{h}_S of the system \mathcal{S} such that $S^\dagger S = S S^\dagger = I$ and H is the Hamiltonian of \mathcal{S} . The equation is written in the interaction picture with respect to free dynamics of the field.

The Hilbert space of the Bose field \mathfrak{h} has a continuous tensor product structure it means that it can be split

it into the “past and future spaces” $\mathfrak{h} = \mathfrak{h}_{[0,t)} \otimes \mathfrak{h}_{[t,+\infty)}$ [14, 16]. In particular, for the vacuum state we have the factorization property

$$|vac\rangle = |vac_{[0,t)}\rangle \otimes |vac_{[t,+\infty)}\rangle.$$

Note that U_t acts non-trivially only in $\mathfrak{h}_S \otimes \mathfrak{h}_{[0,t)}$ and it commutes with the all increments $dB_t = B_{t+dt} - B_t$, $dB_t^\dagger = B_{t+dt}^\dagger - B_t^\dagger$, $d\Lambda_t = \Lambda_{t+dt} - \Lambda_t$ [16]. According to interpretation given by Gardiner and Collet, the operators B_t , B_t^\dagger , Λ_t refer to the input field - the field before interaction with \mathcal{S} . The field after interaction with \mathcal{S} is given by $B_t^{out} = U_t^\dagger B_t U_t$, $B_t^{out\dagger} = U_t^\dagger B_t^\dagger U_t$, $\Lambda_t^{out\dagger} = U_t^\dagger \Lambda_t^\dagger U_t$ and it called the output field [18, 29]. The Bose field operators can be written as

$$B_t = \int_0^t b_s ds, \quad B_t^\dagger = \int_0^t b_s^\dagger ds,$$

$$\Lambda_t = \int_0^t b_s^\dagger b_s ds,$$

where b_t , b_t^\dagger are operators satisfying the canonical commutation rules

$$[b_t, b_s] = 0, \quad [b_t, b_s^\dagger] = \delta(t - s).$$

Note that for the field taken in the vacuum state, we have

$$\langle vac | dB_t | vac \rangle = 0, \quad \langle vac | d\Lambda_t | vac \rangle = 0,$$

$$\langle vac | dB_t dB_t^\dagger | vac \rangle = dt, \quad \langle vac | dB_t^\dagger dB_t | vac \rangle = 0.$$

A. Continuous-mode single photon state

The continuous-mode single photon state is defined as [27, 28]

$$|1_\xi\rangle = \int_0^{+\infty} \xi(t) dB_t^\dagger |vac\rangle$$

with $\xi \in \mathbb{C}$ and has the normalization $\langle 1_\xi | 1_\xi \rangle = \int_0^\infty |\xi(t)|^2 dt = 1$. In the frequency domain it has the form

$$|1_\xi\rangle = \int_{-\infty}^{+\infty} d\omega \tilde{\xi}(\omega) b^\dagger(\omega) |vac\rangle,$$

where $\tilde{\xi}$ is the Fourier transform of ξ .

The mean values of increments dB_t , dB_t^\dagger , $d\Lambda_t$ and their products for the field being in the continuous-mode single photon state are

$$\langle 1_\xi | dB_t | 1_\xi \rangle = \xi(t) dt, \quad \langle 1_\xi | d\Lambda_t | 1_\xi \rangle = |\xi(t)|^2 dt,$$

$$\langle 1_\xi | dB_t dB_t^\dagger | 1_\xi \rangle = dt, \quad \langle 1_\xi | dB_t^\dagger dB_t | 1_\xi \rangle = 0,$$

$$\langle 1_\xi | d\Lambda_t dB_t^\dagger | 1_\xi \rangle = \xi^*(t) dt.$$

Let us notice that for the continuous-mode single photon state we have the additive decomposition property

$$|1_\xi\rangle = |1_{\xi[0,t)}\rangle \otimes |vac_{[t,+\infty)}\rangle + |vac_{[0,t)}\rangle \otimes |1_{\xi[t,+\infty)}\rangle,$$

where

$$|1_{\xi[0,t)}\rangle \otimes |vac_{[t,+\infty)}\rangle = \int_0^t \xi(s) dB_s^\dagger |vac\rangle$$

and

$$|vac_{[0,t)}\rangle \otimes |1_{\xi[t,+\infty)}\rangle = \int_t^{+\infty} \xi(s) dB_s^\dagger |vac\rangle.$$

Taking trace over the space $\mathfrak{h}_{[t,+\infty)}$, we have

$$\begin{aligned} \text{Tr}_{\mathfrak{h}_{[t,+\infty)}} |1_\xi\rangle \langle 1_\xi| &= |1_{\xi[0,t)}\rangle \langle 1_{\xi[0,t)}| \\ &+ |vac_{[0,t)}\rangle \langle vac_{[0,t)}| \int_t^{+\infty} |\xi(s)|^2 ds, \end{aligned}$$

where $\int_t^{+\infty} |\xi(s)|^2 ds$ is the probability that the field is in the vacuum state in the time interval $[0, t)$. Moreover, for any bounded operator of the field having the form $R_t = R_{[0,t)} \otimes I_{[t,+\infty)}$ (a bounded operator acting trivially on $\mathfrak{h}_{[t,+\infty)}$), the mean value for the field being in the continuous-mode single photon state is

$$\begin{aligned} \langle 1_\xi | R_t | 1_\xi \rangle &= \langle 1_{\xi[0,t)} | R_{[0,t)} | 1_{\xi[0,t)} \rangle \\ &+ \langle vac_{[0,t)} | R_{[0,t)} | vac_{[0,t)} \rangle \int_t^{+\infty} |\xi(s)|^2 ds. \end{aligned}$$

B. The reduced dynamics of \mathcal{S}

The operator X of the system \mathcal{S} in the Heisenberg picture, given as

$$j_t(X) = U_t^\dagger (X \otimes I) U_t,$$

is an adapted process, so it acts as the identity in the space $\mathfrak{h}_{[t,+\infty)}$. To derive the differential equation for $j_t(X)$, we apply the rules of the quantum stochastic calculus (QSC) with

$$dj_t(t) = dU_t^\dagger X \otimes IU_t + U_t^\dagger X \otimes IdU_t + dU_t^\dagger X \otimes IdU_t$$

and the Itô table of the form

$$\begin{aligned} dB_t dB_t^\dagger &= dt, \quad dB_t d\Lambda_t = dB_t, \\ d\Lambda_t d\Lambda_t &= d\Lambda_t, \quad d\Lambda_t dB_t^\dagger = dB_t^\dagger \end{aligned}$$

with all the others products, including the products involving dt , vanishing. One can check that for a single-photon state and coherent states, the states which we are interested in, we have the Itô table of that form. Thus, we obtain the stochastic equation

$$\begin{aligned} dj_t(t) &= j_t(\mathcal{L}^* X) dt + j_t(S^\dagger[X, L]) dB_t^\dagger \\ &+ j_t([L^\dagger, X]S) dB_t + j_t(S^\dagger XS - X) d\Lambda_t, \end{aligned}$$

with the superoperator

$$\mathcal{L}^* X = i[H, X] + L^\dagger X L - \frac{1}{2} L^\dagger L X - \frac{1}{2} X L^\dagger L.$$

We assume that the compound system is prepared initially in the state

$$\rho(0) \otimes \rho_{field},$$

where $\rho(0)$ and ρ_{field} are the initial states of \mathcal{S} and the Bose field, respectively. Under this condition the reduced density operator of \mathcal{S} at the time t is given by the formula

$$\rho(t) = \text{Tr}_{\mathfrak{h}} \left(U_t (\rho(0) \otimes \rho_{field}) U_t^\dagger \right).$$

Now, using the property

$$\text{Tr}_{\mathcal{H}_S} (X \rho(t)) = \text{Tr}_{\mathfrak{h}_S \otimes \mathfrak{h}} (j_t(X) \rho(0) \otimes \rho_{field}),$$

one can check that the reduced dynamics of \mathcal{S} for the initial state of the field

$$\begin{aligned} \rho_{field} = & \gamma_{00} |vac\rangle \langle vac| + \gamma_{01} |1_\xi\rangle \langle vac| \\ & + \gamma_{10} |vac\rangle \langle 1_\xi| + \gamma_{11} |1_\xi\rangle \langle 1_\xi| \end{aligned} \quad (1)$$

may be expressed as

$$\rho(t) = \gamma_{00} \rho^{00}(t) + \gamma_{01} \rho^{10}(t) + \gamma_{10} \rho^{01}(t) + \gamma_{11} \rho^{11}(t),$$

where the matrices $\rho^{00}(t)$, $\rho^{10}(t)$, $\rho^{01}(t)$, $\rho^{11}(t)$ satisfy the following system of equations [11]

$$\begin{aligned} \dot{\rho}^{11}(t) &= \mathcal{L} \rho^{11}(t) + [S \rho^{01}(t), L^\dagger] \xi(t) + [L, \rho^{10}(t) S^\dagger] \xi^*(t) \\ &\quad + (S \rho^{00}(t) S^\dagger - \rho^{00}(t)) |\xi(t)|^2, \\ \dot{\rho}^{10}(t) &= \mathcal{L} \rho^{10}(t) + [S \rho^{00}(t), L^\dagger] \xi(t), \\ \dot{\rho}^{01}(t) &= \mathcal{L} \rho^{01}(t) + [L, \rho^{00}(t) S^\dagger] \xi^*(t), \\ \dot{\rho}^{00}(t) &= \mathcal{L} \rho^{00}(t) \end{aligned} \quad (2)$$

with the initial conditions $\rho^{11}(0) = \rho^{00}(0) = \rho(0)$, $\rho^{01}(0) = \rho^{10}(0) = 0$. Here, we have

$$\mathcal{L} \rho = -i[H, \rho] + L \rho L^\dagger - \frac{1}{2} L^\dagger L \rho - \frac{1}{2} \rho L^\dagger L. \quad (3)$$

The choice of the coefficients defining the state (1) is constrained by the conditions that $\rho_{field} \geq 0$, $\rho_{field} = \rho_{field}^\dagger$, and $\text{Tr}\{\rho_{field}\} = 1$. If $\gamma_{00} = 1$ and $\gamma_{11} = \gamma_{01} = \gamma_{10} = 0$, we get the master equation for the vacuum environment and if we take $\gamma_{11} = 1$ and $\gamma_{00} = \gamma_{01} = \gamma_{10} = 0$, we obtain the set of the coupled equations for the external field in a single-photon state. Note that the matrices $\rho^{01}(t)$ and $\rho^{10}(t)$ are non-Hermitian trace-class zero operators and $\rho^{01}(t) = (\rho^{10}(t))^\dagger$.

C. The output processes

The number of photons in the output field in the time interval from t to $t + dt$ is described by the operator

$$d\Lambda_t^{out} = d\Lambda_t + j_t(L^\dagger S) dB_t + j_t(S^\dagger L) dB_t^\dagger + j_t(L^\dagger L) dt.$$

The first term in this expression relates to the incoming photons, the next two terms describe the effect of interaction between the systems, and the last term refers to the photons emitted by \mathcal{S} . The mean value of this operator for the Bose field prepared in (1) is given by

$$\begin{aligned} \langle d\Lambda_t^{out} \rangle = & \gamma_{11} |\xi(t)|^2 dt \\ & + \text{Tr}_{\mathfrak{h}_S} [L^\dagger S (\gamma_{11} \rho^{01}(t) + \gamma_{01} \rho^{00}(t))] \xi(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} [S^\dagger L (\gamma_{11} \rho^{10}(t) + \gamma_{10} \rho^{00}(t))] \xi^*(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} (L^\dagger L) dt. \end{aligned}$$

In homodyne detection we measure the quadrature operator

$$dY(t) = dB_t^{out} + dB_t^{out\dagger},$$

where

$$dB_t^{out} = j_t(S) dB_t + j_t(L) dt.$$

Thus for the Bose field taken in (1), we have the mean

$$\begin{aligned} \langle dY(t) \rangle = & \text{Tr}_{\mathfrak{h}_S} [S (\gamma_{11} \rho^{01}(t) + \gamma_{01} \rho^{00}(t))] \xi(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} [S^\dagger (\gamma_{11} \rho^{10}(t) + \gamma_{10} \rho^{00}(t))] \xi^*(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} [(L + L^\dagger) \rho(t)] dt. \end{aligned}$$

D. The generator of the Bose field in a combination of vacuum and single photon states

As a generator of the Bose field in a combination of vacuum and single photon states, we consider a two level system which interacts with the field in the vacuum state. This system plays for us only a role of ancilla producing a signal in non-classical state. We assume that the evolution of the compound system (ancilla plus the Bose field) is given by the unitary operator, \tilde{U}_t , which satisfies QSDE

$$\begin{aligned} d\tilde{U}_t &= \left(L_A dB_t^\dagger - L_A^\dagger dB_t - \frac{1}{2} L_A^\dagger L_A dt \right) \tilde{U}_t, \quad (4) \\ \tilde{U}_{t=0} &= I, \end{aligned}$$

and we take the coupling operator in the form

$$L_A = \lambda(t) \sigma_-,$$

where $\lambda(t) \in \mathbb{C}$ and σ_- is the lowering operator from the excited $|1\rangle$ to the ground state $|0\rangle$ of ancilla. The Hamiltonian of ancilla $H_A = 0$. The same generator was described in [26]. The Schrödinger equation for the state $|\psi_t\rangle = \tilde{U}_t |\psi_0\rangle \otimes |vac\rangle$ of the compound system then reads

$$d|\psi_t\rangle = \left(\lambda(t) \sigma_- dB_t^\dagger - \frac{1}{2} |\lambda(t)|^2 \sigma_+ \sigma_- dt \right) |\psi_t\rangle. \quad (5)$$

Let us notice that if the coupling coefficient

$$\lambda(t) = \frac{\xi(t)}{\sqrt{\int_t^{+\infty} |\xi(t)|^2 dt}} \quad (6)$$

and if the initial state of the compound system is given by

$$|\psi_{t=0}\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes |vac\rangle$$

then

$$\begin{aligned} |\psi_t\rangle &= c_0|0\rangle \otimes |vac\rangle + c_1|1\rangle \otimes \sqrt{\int_t^{+\infty} |\xi(s)|^2 ds} |vac\rangle \\ &+ c_1|0\rangle \otimes \int_0^t \xi(s) dB_s^\dagger |vac\rangle \end{aligned}$$

is the exact solution of Eq. (5), and

$$\lim_{t \rightarrow +\infty} |\psi_t\rangle = |0\rangle \otimes (c_0|vac\rangle + c_1|1_\xi\rangle).$$

Hence the Bose field after interaction with ancilla till time t (the output field from ancilla) is in the state

$$\begin{aligned} \varrho_{field}^{out}(t) &= |c_0|^2 |vac_{[0,t]}\rangle \langle vac_{[0,t]}| + c_0 c_1^* |vac_{[0,t]}\rangle \langle 1_{\xi[0,t]}| \\ &+ c_0^* c_1 |1_{\xi[0,t]}\rangle \langle vac_{[0,t]}| + |c_1|^2 |1_{\xi[0,t]}\rangle \langle 1_{\xi[0,t]}| \\ &+ |c_1|^2 \int_t^{+\infty} |\xi(s)|^2 ds |vac_{[0,t]}\rangle \langle vac_{[0,t]}|. \end{aligned}$$

In particular, when the ancilla is prepared in the ground state, we get

$$\varrho_{field}^{out}(t) = |vac_{[0,t]}\rangle \langle vac_{[0,t]}|$$

and when ancilla is initially in the excited state, we have

$$\begin{aligned} \varrho_{field}^{out}(t) &= |1_{\xi[0,t]}\rangle \langle 1_{\xi[0,t]}| \\ &+ \int_t^{+\infty} |\xi(s)|^2 ds |vac_{[0,t]}\rangle \langle vac_{[0,t]}| \end{aligned}$$

and

$$\varrho_{field} = |1_\xi\rangle \langle 1_\xi|$$

in the limit $t \rightarrow +\infty$.

Let us consider now the situation when the ancilla is initially in the state

$$\rho_A(0) = \gamma_{00}|0\rangle \langle 0| + \gamma_{10}|0\rangle \langle 1| + \gamma_{01}|1\rangle \langle 0| + \gamma_{11}|1\rangle \langle 1|. \quad (7)$$

One can check that in this case the state of the Bose field after interaction with the ancilla till time t has the form

$$\begin{aligned} \varrho_{field}^{out}(t) &= \gamma_{00}|vac_{[0,t]}\rangle \langle vac_{[0,t]}| + \gamma_{10}|vac_{[0,t]}\rangle \langle 1_{\xi[0,t]}| \\ &+ \gamma_{01}|1_{\xi[0,t]}\rangle \langle vac_{[0,t]}| + \gamma_{11}|1_{\xi[0,t]}\rangle \langle 1_{\xi[0,t]}| \\ &+ \gamma_{11} \int_t^{+\infty} |\xi(s)|^2 ds |vac_{[0,t]}\rangle \langle vac_{[0,t]}| \end{aligned}$$

and in the limit $t \rightarrow +\infty$, we get

$$\begin{aligned} \varrho_{field} &= \gamma_{00}|vac\rangle \langle vac| + \gamma_{10}|vac\rangle \langle 1_\xi| \\ &+ \gamma_{01}|1_\xi\rangle \langle vac| + \gamma_{11}|1_\xi\rangle \langle 1_\xi|. \end{aligned}$$

Let us note that the expression (6) we derived under the assumption that the denominator is different from zero. The relation between $\xi(t)$ and $\lambda(t)$ implies that $\lambda(t) = 0$ whenever $\xi(t) = 0$.

E. Master equation for the extended system

In cascaded quantum systems the output of the first system is supplied into the second system, but the reverse process is forbidden. We consider a cascaded system consisting of the ancilla producing the field in the desired state and the system \mathcal{S} . The ancilla system is driven by the Bose field in the vacuum state and the output from the ancilla becomes the input field for \mathcal{S} . When we omit a time shift due to a traveling between the ancilla and \mathcal{S} , we get the master equation for the extended system of the form [18]

$$\begin{aligned} \dot{\tilde{\rho}}(t) &= \mathcal{L}\tilde{\rho}(t) + \mathcal{L}_A\tilde{\rho}(t) + [SL_A\tilde{\rho}(t), L^\dagger] \\ &+ [L, \tilde{\rho}(t)L_A^\dagger S^\dagger] \\ &+ (SL_A\tilde{\rho}(t)L_A^\dagger S^\dagger - L_A\tilde{\rho}(t)L_A^\dagger), \end{aligned} \quad (8)$$

where

$$\mathcal{L}_A\tilde{\rho} = L_A\tilde{\rho}L_A^\dagger - \frac{1}{2}L_A^\dagger L_A\tilde{\rho} - \frac{1}{2}\tilde{\rho}L_A^\dagger L_A.$$

We make the assumption that the extended system is initially prepared in the uncorrelated state

$$\tilde{\rho}(t=0) = \rho(0) \otimes \rho_A(0),$$

where $\rho_A(0)$ is the initial state of ancilla given by (7).

Now by taking the partial trace of both sides of (8) over the Hilbert space of ancilla, we obtain the equation

$$\begin{aligned} \dot{\rho}_S(t) &= \mathcal{L}\rho_S(t) + [S \text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t)), L^\dagger] \\ &+ [L, \text{Tr}_{\mathfrak{h}_A}(\tilde{\rho}(t)L_A^\dagger) S^\dagger] \\ &+ (S \text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t)L_A^\dagger) S^\dagger \\ &- \text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t)L_A^\dagger)) \end{aligned} \quad (9)$$

for $\rho_S(t) = \text{Tr}_{\mathfrak{h}_A}\tilde{\rho}(t)$, which describes the reduced dynamics of \mathcal{S} .

To determine the solution of the above equation, we need to find also the operators $\text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t))$, $\text{Tr}_{\mathfrak{h}_A}(\tilde{\rho}(t)L_A^\dagger)$, and $\text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t)L_A^\dagger)$. For this purpose we derive the differential equations for the operators defined by

$$\begin{aligned} \rho_S^-(t) &= \frac{1}{\xi(t)} \text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t)) \\ &= \frac{1}{\sqrt{\int_t^{+\infty} |\xi(s)|^2 ds}} \text{Tr}_{\mathfrak{h}_A}(\sigma_- \tilde{\rho}(t)), \end{aligned}$$

$$\begin{aligned} \rho_S^\mp(t) &= \frac{1}{|\xi(t)|^2} \text{Tr}_{\mathfrak{h}_A}(L_A\tilde{\rho}(t)L_A^\dagger) \\ &= \frac{1}{\int_t^{+\infty} |\xi(s)|^2 ds} \text{Tr}_{\mathfrak{h}_A}(\sigma_- \tilde{\rho}(t) \sigma_+). \end{aligned}$$

If $\xi(t) = 0$, we put simply $\rho_S^-(t) = \rho_S^+(t) = \rho_S^\mp(t) = 0$. It is not difficult to check that they satisfy the following set of equations:

$$\dot{\rho}_S^-(t) = \mathcal{L}\rho_S^-(t) + [L, \rho_S^\mp(t)S^\dagger] \xi^*(t),$$

$$\dot{\rho}_S^\mp(t) = \mathcal{L}\rho_S^\mp(t)$$

with the initial conditions $\rho_S^\mp(t) = \gamma_{11}\rho(0)$, $\rho_S^-(0) = \gamma_{01}\rho(0)$, and $(\rho_S^-(t))^\dagger = \rho_S^+(t)$. Thus to specify the reduced state of \mathcal{S} , we have to solve the set of differential equations for $\rho_S(t)$, $\rho_S^-(t)$, $\rho_S^+(t)$. Note that for all $t \geq 0$ $\text{Tr}_{\mathfrak{h}_S}\rho_S(t) = 1$ and the traces of the matrices $\rho_S^-(t)$, $\rho_S^+(t)$, and $\rho_S^\mp(t)$ are respectively equal to γ_{01} , γ_{10} , and γ_{11} .

The reduced dynamics given by (9) is equivalent to the reduced dynamics described in Part II.B. To show this one has to write the differential equation for $\varrho(t)$ by adding Eqs. (2) with suitable coefficients: γ_{11} , γ_{01} , γ_{10} , and γ_{00} . Then one can see that: $\rho_S^-(t) = \gamma_{11}\rho^{01}(t) + \gamma_{01}\rho^{00}(t)$, $\rho_S^+(t) = \gamma_{11}\rho^{10}(t) + \gamma_{10}\rho^{00}(t)$, $\rho_S^\mp(t) = \gamma_{11}\rho^{00}(t)$, and $\varrho(t) = \varrho_S(t)$, which ends the proof.

The expressions for the mean values of photons in the interval t to $t + dt$

$$\begin{aligned} \langle d\Lambda_t^{out} \rangle = & \text{Tr}_{\mathfrak{h}_S} (L^\dagger L \rho_S(t)) dt + \text{Tr}_{\mathfrak{h}_S} (L^\dagger S \rho_S^-(t)) \xi(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} (S^\dagger L \rho_S^+(t)) \xi^*(t) dt + \gamma_{11} |\xi(t)|^2 dt \end{aligned}$$

and for the mean value of the optical quadrature in the interval t to $t + dt$

$$\begin{aligned} \langle dY(t) \rangle = & \text{Tr}_{\mathfrak{h}_S} (S \rho_S^-(t)) \xi(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} (S^\dagger \rho_S^+(t)) \xi^*(t) dt \\ & + \text{Tr}_{\mathfrak{h}_S} ((L^\dagger + L) \rho_S(t)) dt \end{aligned}$$

obtained in the model of cascaded systems are in agreement with the formulas derived in Part II.C.

F. The quantum trajectories for the photon counting

Quantum filtering equation for a system coupled to the Bose field in the vacuum is well known and we can easily write it down for the extended system consisting of the ancilla and \mathcal{S} . The stochastic master equation for the extended system and the direct observation of photons leaving the system has the form

$$\begin{aligned} d\hat{\rho}(t) = & \mathcal{L}\hat{\rho}(t)dt + \mathcal{L}_A\hat{\rho}(t)dt + [SL_A\hat{\rho}(t), L^\dagger]dt \\ & + [L, \hat{\rho}(t)L_A^\dagger S^\dagger]dt \\ & + \left(SL_A\hat{\rho}(t)L_A^\dagger S^\dagger - L_A\hat{\rho}(t)L_A^\dagger \right) dt \\ & + \left\{ \frac{(L + SL_A)\hat{\rho}(t)(L + SL_A)^\dagger}{K_t} - \hat{\rho}(t) \right\} dN(t) \end{aligned}$$

where

$$dN(t) = d\Lambda_t^{out} - K_t dt,$$

and

$$K_t = \text{Tr}_{\mathfrak{h}_S \otimes \mathfrak{h}_A} [(L + SL_A)^\dagger (L + SL_A) \hat{\rho}(t)].$$

Eq. (10) describes the evolution of the extended system conditioned by the trajectory of all past results till the time t . Here $d\Lambda_t^{out}$ is the output process for the cascaded system with the posterior mean value equal to $K_t dt$. In other words, the quantity $K_t dt$ is the mean value of counts in the interval t to $t + dt$ conditional upon the trajectory (the history of all counts) up to time t . Note that $(dN(t))^2 = dN(t)$ and the mean $\langle dN(t) \rangle = 0$, which follows from the properties of $d\Lambda_t^{out}$.

Using the characteristic functional method [16], we can find the whole statistics of the output counting process. Let us recall that we deal with the regular counting process which means that only one photon can be observed at a given instant of time. The probability of having no count in the time-interval $(0, t]$ is given by the expression

$$P_0^t(0) = \text{Tr}_{\mathfrak{h}_S \otimes \mathfrak{h}_A} [\Upsilon(t, 0) \rho(0) \otimes \rho_A(0)], \quad (11)$$

where $\Upsilon(t, s)$, $t \geq s$, is defined by

$$\frac{d}{dt} \Upsilon(t, s) = \tilde{\mathcal{L}}_t \Upsilon(t, s)$$

with the condition $\Upsilon(s, s) = 1$, and

$$\tilde{\mathcal{L}}_t \rho = -iH_{\text{eff}} \rho + i\rho H_{\text{eff}}^\dagger,$$

where H_{eff} is the effective Hamiltonian having the form

$$H_{\text{eff}} = H_S - \frac{i}{2} (L^\dagger L + L_A^\dagger L_A + 2L_A L^\dagger S).$$

One can check that $P_0^{t=0}(0) = 1$. Of course, the quantity $1 - P_0^t(0)$ is the probability of at least one count in the interval $(0, t]$. The multi-time probability density of a count at time t_1 , a count at time t_2 , ..., $(0 < t_1 < t_2 < \dots < t_n < t)$ and no other counts in the interval from 0 to t is given by

$$\begin{aligned} p_0^0(t_1; t_2, \dots, t_n) = & \text{Tr}_{\mathfrak{h}_S \otimes \mathfrak{h}_A} [\Upsilon(t, t_n) \mathcal{J}(t_n) \Upsilon(t_n, t_{n-1}) \dots \\ & \dots \Upsilon(t_2, t_1) \mathcal{J}(t_1) \Upsilon(t_1, 0) \rho(0) \otimes \rho_A(0)], \end{aligned}$$

where

$$\mathcal{J}(t_i) \rho = (L + SL_A(t_i)) \rho (L + SL_A(t_i))^\dagger.$$

Our notation in the above expression indicates the fact that the operator L_A depends on time.

Now taking the partial trace of Eq. (10) over the Hilbert space of ancilla, we obtain the filtering equation

for the system \mathcal{S} :

$$\begin{aligned} d\hat{\rho}_S(t) = & \mathcal{L}\hat{\rho}_S(t)dt + [S\hat{\rho}_S^-(t), L^\dagger]\xi(t)dt \\ & + [L, \hat{\rho}_S^+(t)S^\dagger]\xi^*(t)dt \\ & + (S\hat{\rho}_S^\mp(t)S^\dagger - \hat{\rho}_S^\mp(t))|\xi(t)|^2dt \\ & + \left\{ \frac{1}{K_t} [L\hat{\rho}_S(t)L^\dagger + S\hat{\rho}_S^-(t)L^\dagger\xi(t) \right. \\ & + L\hat{\rho}_S^+(t)S^\dagger\xi^*(t) \\ & \left. + S\hat{\rho}_S^\mp(t)S^\dagger|\xi(t)|^2] - \hat{\rho}_S(t) \right\} dN(t) \end{aligned} \quad (12)$$

where the operators

$$\hat{\rho}_S^-(t) = \frac{1}{\xi(t)} \text{Tr}_{\mathfrak{h}_A} (L_A \hat{\rho}(t)) = \frac{\text{Tr}_{\mathfrak{h}_A} (\sigma_- \hat{\rho}(t))}{\sqrt{\int_t^{+\infty} |\xi(s)|^2 ds}},$$

$$\hat{\rho}_S^\mp(t) = \frac{1}{|\xi(t)|^2} \text{Tr}_{\mathfrak{h}_A} (L_A \hat{\rho}(t) L_A^\dagger) = \frac{\text{Tr}_{\mathfrak{h}_A} (\sigma_- \hat{\rho}(t) \sigma_+)}{\int_t^{+\infty} |\xi(s)|^2 ds}$$

satisfy the set of the coupled stochastic equations

$$\begin{aligned} d\hat{\rho}_S^-(t) = & \mathcal{L}\hat{\rho}_S^-(t)dt + [L, \hat{\rho}_S^\mp(t)S^\dagger]\xi^*(t)dt \\ & + \left\{ \frac{1}{K_t} (L\hat{\rho}_S^-(t)L^\dagger + L\hat{\rho}_S^\mp(t)S^\dagger\xi^*(t)) \right. \\ & \left. - \hat{\rho}_S^-(t) \right\} dN(t), \end{aligned}$$

$$\begin{aligned} d\hat{\rho}_S^\mp(t) = & \mathcal{L}\hat{\rho}_S^\mp(t)dt \\ & + \left(\frac{1}{K_t} L\hat{\rho}_S^\mp(t)L^\dagger - \hat{\rho}_S^\mp(t) \right) dN(t), \end{aligned}$$

with the initial condition $\hat{\rho}_S^\mp(0) = \gamma_{11}\rho(0)$, $\hat{\rho}_S^-(0) = \gamma_{01}\rho(0)$, and $\hat{\rho}_S(0) = \rho(0)$. The posterior mean value K_t can be thus written as

$$\begin{aligned} K_t = & \text{Tr}_{\mathfrak{h}_S} [L^\dagger L \hat{\rho}_S(t) + L^\dagger S \hat{\rho}_S^-(t) \xi(t) \\ & + S^\dagger L \hat{\rho}_S^+(t) \xi^*(t) + |\xi(t)|^2 \text{Tr}_{\mathfrak{h}_S} \hat{\rho}_S^\mp(t)]. \end{aligned}$$

Eq. (11) implies that the probability of having no counts in the time interval from 0 to t can be expressed as $P_0^t(0) = \text{Tr}_{\mathfrak{h}_S} \tilde{\rho}_S(t)$ with $\tilde{\rho}_S(t)$ that we obtain by solving the set of equations

$$\begin{aligned} \dot{\tilde{\rho}}_S(t) = & -i[H_S, \tilde{\rho}_S(t)] - \frac{1}{2} L^\dagger L \tilde{\rho}_S(t) - \frac{1}{2} \tilde{\rho}_S(t) L^\dagger L \\ & - L^\dagger S \tilde{\rho}_S^-(t) \xi(t) - \tilde{\rho}_S^+(t) S^\dagger L \xi^*(t) \\ & - \tilde{\rho}_S^\mp(t) |\xi(t)|^2, \end{aligned} \quad (13)$$

$$\begin{aligned} \dot{\tilde{\rho}}_S^-(t) = & -i[H_S, \tilde{\rho}_S^-(t)] - \frac{1}{2} L^\dagger L \tilde{\rho}_S^-(t) - \frac{1}{2} \tilde{\rho}_S^-(t) L^\dagger L \\ & - \tilde{\rho}_S^\mp(t) S^\dagger L \xi^*(t), \end{aligned} \quad (14)$$

$$\dot{\tilde{\rho}}_S^\mp(t) = -i[H_S, \tilde{\rho}_S^\mp(t)] - \frac{1}{2} L^\dagger L \tilde{\rho}_S^\mp(t) - \frac{1}{2} \tilde{\rho}_S^\mp(t) L^\dagger L \quad (15)$$

with $\tilde{\rho}_S^\mp(0) = \gamma_{11}\rho(0)$, $\tilde{\rho}_S^-(0) = \gamma_{01}\rho(0)$, and $\tilde{\rho}_S(0) = \rho(0)$.

We found the conditional evolution of \mathcal{S} depending on the results of measurement performed on the output Bose field. If these results are not read, i.e. no selection is made, the state of the system \mathcal{S} at time t is given by $\rho_S(t)$ which fulfills Eq. (9). We can derive the master equation for \mathcal{S} by taking the stochastic mean of Eq. (??).

Note that when $\gamma_{01} = \gamma_{10} = \gamma_{11} = 0$ and $\gamma_{00} = 1$, we get $\hat{\rho}_S^\mp(t) = 0$ and $\hat{\rho}_S^-(t) = 0$ for all t and Eq. (12) reduces then to the filtering equation for the Bose field in the vacuum state. When $\gamma_{01} = \gamma_{10} = \gamma_{00} = 0$ and $\gamma_{11} = 1$, we obtain the Bose field in a single photon state and our filtering equation reduces to the filtering equation derived in [25, 26].

G. The quantum trajectories for the quadrature measurement

The conditional evolution of the extended system for the case of the quadrature measurement of the output field is given by

$$\begin{aligned} d\hat{\rho}(t) = & \mathcal{L}\hat{\rho}(t)dt + \mathcal{L}_A \hat{\rho}(t)dt + [SL_A \hat{\rho}(t), L^\dagger]dt \\ & + [L, \hat{\rho}(t) L_A^\dagger S^\dagger]dt \\ & + (SL_A \tilde{\rho}(t) L_A^\dagger S^\dagger - L_A \tilde{\rho}(t) L_A^\dagger) dt \\ & + ((L + SL_A) \hat{\rho}(t) + \hat{\rho}(t) (L + SL_A)^\dagger \\ & - K_t \hat{\rho}(t)) dW(t), \end{aligned} \quad (16)$$

where

$$K_t = \text{Tr}_{\mathfrak{h}_S \otimes \mathfrak{h}_A} [(L + L^\dagger + SL_A + L_A^\dagger S^\dagger) \hat{\rho}(t)]$$

and

$$dW(t) = dY(t) - K_t dt$$

is the innovation Wiener process, where $dY(t) = dB_t^{out} + dB_t^{out\dagger}$ is the output field from the cascaded system. The equation (16) sets the posterior state $\hat{\rho}(t)$ of the extended system at time t depending on the stochastic trajectory up to time t . Note that the posterior means $\langle dY(t) \rangle = K_t dt$, $\langle dW(t) \rangle = 0$, and $\langle (dW(t))^2 \rangle = dt$.

The partial trace over the Hilbert space of ancilla allows us to derive from (16) the stochastic evolution of \mathcal{S} . The filtering equation for the conditional state of \mathcal{S} , $\hat{\rho}_S(t) = \text{Tr}_{\mathfrak{h}_A} \hat{\rho}(t)$, has thus the form

$$\begin{aligned} d\hat{\rho}_S(t) = & \mathcal{L}\hat{\rho}_S(t)dt + [S\hat{\rho}_S^-(t), L^\dagger]\xi(t)dt \\ & + [L, \hat{\rho}_S^+(t)S^\dagger]\xi^*(t)dt \\ & + (S\hat{\rho}_S^\mp(t)S^\dagger - \hat{\rho}_S^\mp(t))|\xi(t)|^2dt \\ & + (L\hat{\rho}_S(t) + \hat{\rho}_S(t)L^\dagger + S\hat{\rho}_S^-(t)\xi(t) \\ & + \hat{\rho}_S^+(t)S^\dagger\xi^*(t) - K_t \hat{\rho}_S(t)) dW(t), \end{aligned} \quad (17)$$

where the matrices satisfy the equations

$$\begin{aligned} d\hat{\rho}_S^-(t) = & \mathcal{L}\hat{\rho}_S^-(t)dt + [L, \hat{\rho}_S^-(t)S] \xi^*(t)dt \\ & + (L\hat{\rho}_S^-(t) + \hat{\rho}_S^-(t)L^\dagger \\ & + \hat{\rho}_S^-(t)S^\dagger \xi^*(t) - K_t \hat{\rho}_S^-(t)) dW(t), \end{aligned}$$

$$\begin{aligned} d\hat{\rho}_S^\mp(t) = & \mathcal{L}\hat{\rho}_S^\mp(t)dt \\ & + (L\hat{\rho}_S^\mp(t) + \hat{\rho}_S^\mp(t)L^\dagger - K_t \hat{\rho}_S^\mp(t)) dW(t), \end{aligned}$$

and initially $\hat{\rho}_S(0) = \rho_S(0)$, $\hat{\rho}_S^-(0) = \gamma_{11}\rho_S(0)$, and $\hat{\rho}_S^+(0) = \gamma_{01}\rho_S(0)$. One can check moreover that

$$K_t = \text{Tr}_{\mathfrak{h}_S}[(L + L^\dagger)\hat{\rho}_S(t) + S\hat{\rho}_S^-(t)\xi(t) + \hat{\rho}_S^+(t)S^\dagger\xi^*(t)].$$

H. Numerical example

As numerical example we consider the system \mathcal{S} being a two level atom. We take the operators $L = \sqrt{\kappa}\sigma_-$, $S = I$, and the Hamiltonian $H_S = 0$ (it means that we work in the interaction picture). We describe the case when the Bose field is characterized by

$$\xi(t) = f(t) \left(\int_0^{+\infty} f(s)^2 ds \right)^{-1/2},$$

where

$$f(t) = \left(\frac{\Omega^2}{2\pi} \right)^{1/4} \exp \left[-\frac{\Omega^2}{4} (t-3)^2 \right].$$

To get an optimal excitation we choose the Gaussian pulse with $\Omega = 1.46\kappa$ and put $\kappa = 1$ [8]. The quantum filter is given here by the set of eleven equations.

We plot the curve showing the dependence of time for the probability of being in the excited state for the two level atom and the probability of at least one count in the interval $(0, t]$. The probability of at least one count in the interval from 0 to $+\infty$ is given by the formula $1 - \langle 0|\rho(0)|0\rangle(1 - \gamma_{11})$ which can be deduced from Eqs. (13)-(15).

III. FILTERING EQUATION FOR SYSTEM DRIVEN BY THE BOSE FIELD IN A MIXTURE OF COHERENT STATES

A. Continuous-mode coherent states

Continuous-mode coherent state is defined as

$$|\alpha\rangle = \overrightarrow{T} \exp \left\{ \int_0^{+\infty} \alpha(t) dB_t^\dagger - \alpha^*(t) dB_t \right\} |vac\rangle,$$

where \overrightarrow{T} stands for the chronological ordering operator. Note that

$$\begin{aligned} \overrightarrow{T} \exp \left\{ \int_0^t \alpha(s) dB_s^\dagger - \alpha^*(s) dB_s \right\} |vac\rangle = \\ = |\alpha_{[0,t]}\rangle \otimes |vac_{[t,+\infty)}\rangle, \end{aligned}$$

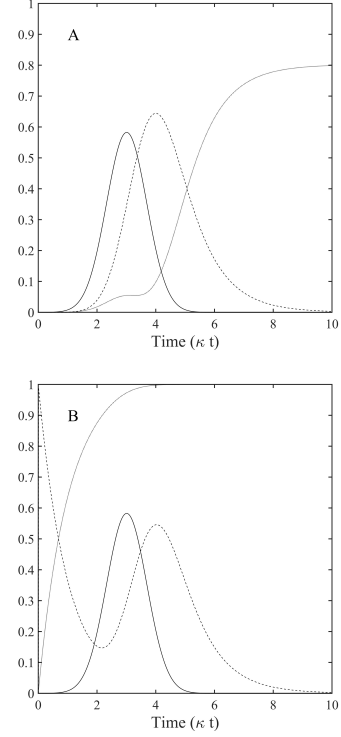


FIG. 1: The solid line is $|\xi(t)|^2$, the dashed line is the excitation calculated from the master equation, and the dotted line is the probability of at least one count in the interval $(0, t]$ for the Bose field taken in a mixture of vacuum and single photon states with $\gamma_{11} = \gamma_{00} = 0.5$, $\gamma_{01} = \gamma_{10} = 0$, and for the two-level system being initially in the ground state (A) and in the upper state (B).

so that the continuous-mode coherent state has the factorization property

$$|\alpha\rangle = |\alpha_{[0,t]}\rangle \otimes |\alpha_{[t,+\infty)}\rangle.$$

The mean values of increments dB_t , dB_t^\dagger , $d\Lambda_t$ for the coherent state are

$$\langle \alpha | dB_t | \alpha \rangle = \alpha(t)dt, \quad \langle \alpha | dB_t^\dagger | \alpha \rangle = \alpha^*(t)dt,$$

$$\langle \alpha | d\Lambda_t | \alpha \rangle = |\alpha(t)|^2 dt.$$

B. The reduced dynamics of \mathcal{S}

If the input Bose field is in the state

$$\rho_{field} = p|\alpha_0\rangle\langle\alpha_0| + (1-p)|\alpha_1\rangle\langle\alpha_1|$$

with $p \in [0, 1]$, then the reduced evolution of the open quantum system interacting with such field can be written as

$$\rho(t) = p\rho^{00}(t) + (1-p)\rho^{11}(t)$$

with the matrices $\rho^{00}(t)$ and $\rho^{11}(t)$ which satisfy the following set of equations

$$\dot{\rho}^{00}(t) = \mathcal{L}\rho^{00}(t) + \alpha_0(t) [S\rho^{00}(t), L^\dagger] + \alpha_0^*(t) [L, \rho^{00}(t)S^\dagger] + |\alpha_0(t)|^2 (S\rho^{00}(t)S^\dagger - \rho^{00}(t)),$$

$$\dot{\rho}^{11}(t) = \mathcal{L}\rho^{11}(t) + \alpha_1(t) [S\rho^{11}(t), L^\dagger] + \alpha_1^*(t) [L, \rho^{11}(t)S^\dagger] + |\alpha_1(t)|^2 (S\rho^{11}(t)S^\dagger - \rho^{11}(t))$$

with the initial conditions $\rho^{00}(0) = \rho^{11}(0) = \rho(0)$ and the superoperator \mathcal{L} given by (3).

C. The generator of the Bose field in a mixture of two coherent states

Let us consider a two level system interacting with the Bose field in the vacuum state. The unitary evolution of the ancilla and the Bose field is given by Eq. (4), but this time we assume that the coupling operator is defined as

$$L_A = \alpha_0(t)|0\rangle\langle 0| + \alpha_1(t)|1\rangle\langle 1|, \quad (18)$$

where $|0\rangle, |1\rangle$ is the orthonormal basis of ancilla.

So that if the initial state of the compound system is

$$(c_0|0\rangle + c_1|1\rangle) \otimes |vac\rangle,$$

then the system evolves according to the formula

$$|\psi_t\rangle = c_0|0\rangle \otimes |\alpha_{0[0,t]}\rangle \otimes |vac_{[t,+\infty)}\rangle + c_1|1\rangle \otimes |\alpha_{1[0,t]}\rangle \otimes |vac_{[t,+\infty)}\rangle$$

and in the limit we have

$$\lim_{t \rightarrow +\infty} |\psi_t\rangle = c_0|0\rangle \otimes |\alpha_0\rangle + c_1|1\rangle \otimes |\alpha_1\rangle.$$

The state of the Bose field after interaction with ancilla till time t has the form

$$\varrho_{field}^{out}(t) = |c_0|^2 |\alpha_{0[0,t]}\rangle\langle \alpha_{0[0,t]}| + |c_1|^2 |\alpha_{1[0,t]}\rangle\langle \alpha_{1[0,t]}|.$$

This shows that choosing the initial state of the ancilla as

$$\rho_A(0) = |c_0|^2 |0\rangle\langle 0| + c_0 c_1^* |0\rangle\langle 1| + c_0^* c_1 |1\rangle\langle 0| + |c_1|^2 |1\rangle\langle 1| \quad (19)$$

and the coupling operator in the form of (18), we obtain the output field in a mixture of two coherent states.

The reduced dynamics of the system coupled to the ancilla generating the Bose field in a mixture of two coherent states, we obtain from Eq. (8) taking the partial trace over ancilla's degrees of freedom. It is not difficult to check that in the considered situation the reduced dynamics of \mathcal{S} is given by

$$\rho_S(t) = \rho_S^{00}(t) + \rho_S^{11}(t),$$

where

$$\rho_S^{00}(t) = \langle 0|\hat{\rho}(t)|0\rangle, \quad \rho_S^{11}(t) = \langle 1|\hat{\rho}(t)|1\rangle$$

satisfy the differential equations

$$\begin{aligned} \dot{\rho}_S^{00}(t) &= \mathcal{L}\rho_S^{00}(t) + \alpha_0(t) [\rho_S^{00}(t), L^\dagger] \\ &\quad + \alpha_0^*(t) [L, \rho_S^{00}(t)] \\ &\quad + |\alpha_0(t)|^2 (S\rho_S^{00}(t)S^\dagger - \rho_S^{00}(t)), \end{aligned}$$

$$\begin{aligned} \dot{\rho}_S^{11}(t) &= \mathcal{L}_S \rho_S^{11}(t) + \alpha_1(t) [\rho_S^{11}(t), L_S^\dagger] \\ &\quad + \alpha_1^*(t) [L_S, \rho_S^{11}(t)] \\ &\quad + |\alpha_1(t)|^2 (S\rho_S^{11}(t)S^\dagger - \rho_S^{11}(t)) \end{aligned}$$

and $\rho_S^{00}(0) = |c_0|^2 \rho_S(0)$, $\rho_S^{11}(0) = |c_1|^2 \rho_S(0)$.

It is seen that when $|c_0|^2 = p$ and $|c_1|^2 = 1 - p$, the above procedure gives the same reduced dynamics of \mathcal{S} as we deal with in Part III. B.

D. The quantum trajectories

Our starting point for derivation of the stochastic evolution of \mathcal{S} is the filtering equation for the extended system consisting of the ancilla and \mathcal{S} . We consider two kinds of conditional evolution, namely, the evolutions conditioned on the results of the direct counting of the photons and on the measurement of the optical quadrature of the output field. The filtering equation for the extended system is given then respectively by Eq. (10) and Eq. (16), but we take L_A in the form of (18) and the initial state of ancilla (19).

The conditional state of \mathcal{S} may now be expressed by

$$\hat{\rho}_S(t) = \hat{\rho}_S^{00}(t) + \hat{\rho}_S^{11}(t),$$

where the matrices

$$\hat{\rho}_S^{00}(t) = \langle 0|\hat{\rho}(t)|0\rangle, \quad \hat{\rho}_S^{11}(t) = \langle 1|\hat{\rho}(t)|1\rangle$$

and $\hat{\rho}(t)$ is the conditional state of the extended system, and initially we take $\hat{\rho}_S^{00}(0) = p\rho(0)$, $\hat{\rho}_S^{11}(0) = (1-p)\rho(0)$.

The matrices $\hat{\rho}_S^{00}(t)$, $\hat{\rho}_S^{11}(t)$ for the photon counting process satisfy the set of stochastic equations

$$\begin{aligned} d\hat{\rho}_S^{00}(t) &= \mathcal{L}\hat{\rho}_S^{00}(t)dt + \alpha_0(t) [S\hat{\rho}_S^{11}(t), L^\dagger] dt \\ &\quad + \alpha_0^*(t) [L, \hat{\rho}_S^{00}(t)S^\dagger] dt \\ &\quad + |\alpha_0(t)|^2 (S\hat{\rho}_S^{00}(t)S^\dagger - \hat{\rho}_S^{00}(t)) dt \\ &\quad + \left\{ \frac{1}{K_t} (L\hat{\rho}_S^{00}(t)L^\dagger + \alpha_0^*(t)L\hat{\rho}_S^{00}(t)S^\dagger \right. \\ &\quad \left. + \alpha_0(t)S\hat{\rho}_S^{00}(t)L^\dagger + |\alpha_0(t)|^2 S\hat{\rho}_S^{00}(t)S^\dagger \right. \\ &\quad \left. - \hat{\rho}_S^{00}(t) \right\} dN(t), \end{aligned}$$

$$\begin{aligned}
d\hat{\rho}_S^{11}(t) = & \mathcal{L}\hat{\rho}_S^{11}(t)dt + \alpha_1(t) [S\hat{\rho}_S^{11}(t), L^\dagger] dt \\
& + \alpha_1^*(t) [L, \hat{\rho}_S^{11}(t)S^\dagger] dt \\
& + |\alpha_1(t)|^2 (S\hat{\rho}_S^{11}(t)S^\dagger - \hat{\rho}_S^{11}(t)) dt \\
& + \left\{ \frac{1}{K_t} (L\hat{\rho}_S^{11}(t)L^\dagger + \alpha_1^*(t)L\hat{\rho}_S^{11}(t)S^\dagger \right. \\
& + \alpha_1(t)S\hat{\rho}_S^{11}(t)L^\dagger + |\alpha_1(t)|^2 S\hat{\rho}_S^{11}(t)S^\dagger) \\
& \left. - \hat{\rho}_S^{11}(t) \right\} dN(t).
\end{aligned}$$

We deduced it by taking the trace over ancilla in Eq. (10). Here $dN(t) = d\Lambda_t^{out} - K_t dt$, where $d\Lambda_t^{out}$ describes the output number of photons for the cascaded system in the interval from t to $t + dt$ with the posterior mean value

$$\begin{aligned}
\langle d\Lambda_t^{out} \rangle = & K_t dt = \text{Tr}_{\mathfrak{h}_S} \{ L^\dagger L \hat{\rho}_S(t) \\
& + L^\dagger S(\alpha_0(t)\hat{\rho}_S^{00}(t) + \alpha_1(t)\hat{\rho}_S^{11}(t)) \\
& + S^\dagger L(\alpha_0^*(t)\hat{\rho}_S^{00}(t) + \alpha_1^*(t)\hat{\rho}_S^{11}(t)) \\
& + |\alpha_0(t)|^2 \hat{\rho}_S^{00}(t) + |\alpha_1(t)|^2 \hat{\rho}_S^{11}(t) \} dt.
\end{aligned}$$

The probability of having at least one count in the interval 0 to t is given in this case by $P_0^t(0) = \text{Tr}_{\mathfrak{h}_S} (\hat{\rho}_S^{00}(t) + \hat{\rho}_S^{11}(t))$, where

$$\begin{aligned}
\dot{\hat{\rho}}_S^{00}(t) = & -i[H_S, \hat{\rho}_S^{00}(t)] - \frac{1}{2}L^\dagger L \hat{\rho}_S^{00}(t) - \frac{1}{2}\hat{\rho}_S^{00}(t)L^\dagger L \\
& - L^\dagger S \hat{\rho}_S^{00}(t)\alpha_0(t) - \hat{\rho}_S^{00}(t)S^\dagger L\alpha_0^*(t) \\
& - \hat{\rho}_S^{00}(t)|\alpha_0(t)|^2, \\
\dot{\hat{\rho}}_S^{11}(t) = & -i[H_S, \hat{\rho}_S^{11}(t)] - \frac{1}{2}L^\dagger L \hat{\rho}_S^{11}(t) - \frac{1}{2}\hat{\rho}_S^{11}(t)L^\dagger L \\
& - L^\dagger S \hat{\rho}_S^{11}(t)\alpha_1(t) - \hat{\rho}_S^{11}(t)S^\dagger L\alpha_1^*(t) \\
& - \hat{\rho}_S^{11}(t)|\alpha_1(t)|^2,
\end{aligned}$$

and $\hat{\rho}_S^{00}(0) = p\rho(0)$, $\hat{\rho}_S^{11}(0) = (1-p)\rho(0)$.

The a posteriori state of \mathcal{S} for the measurement of the optical quadrature is given by $\hat{\rho}_S(t)$ and the matrices $\hat{\rho}_S^{00}(t)$, $\hat{\rho}_S^{11}(t)$ one can find by solving the set of stochastic equations

$$\begin{aligned}
d\hat{\rho}_S^{00}(t) = & \mathcal{L}\hat{\rho}_S^{00}(t)dt + \alpha_0(t) [S\hat{\rho}_S^{11}(t), L^\dagger] dt \\
& + \alpha_0^*(t) [L, \hat{\rho}_S^{00}(t)S^\dagger] dt \\
& + |\alpha_0(t)|^2 (S\hat{\rho}_S^{00}(t)S^\dagger - \hat{\rho}_S^{00}(t)) dt \\
& + \{ L\hat{\rho}_S^{00}(t) + \hat{\rho}_S^{00}(t)L^\dagger + \alpha_0(t)S\hat{\rho}_S^{00}(t) \\
& + \alpha_0^*(t)\hat{\rho}_S^{00}(t)S^\dagger - K_t\hat{\rho}_S^{00}(t) \} dW(t), \\
d\hat{\rho}_S^{11}(t) = & \mathcal{L}\hat{\rho}_S^{11}(t)dt + \alpha_1(t) [S\hat{\rho}_S^{11}(t), L^\dagger] dt \\
& + \alpha_1^*(t) [L, \hat{\rho}_S^{11}(t)S^\dagger] dt \\
& + |\alpha_1(t)|^2 (S\hat{\rho}_S^{11}(t)S^\dagger - \hat{\rho}_S^{11}(t)) dt \\
& + \{ L\hat{\rho}_S^{11}(t) + \hat{\rho}_S^{11}(t)L^\dagger + \alpha_1(t)S\hat{\rho}_S^{11}(t) \\
& + \alpha_1^*(t)\hat{\rho}_S^{11}(t)S^\dagger - K_t\hat{\rho}_S^{11}(t) \} dW(t)
\end{aligned}$$

with the initial conditions $\hat{\rho}_S^{00}(0) = |c_0|^2 \rho_S(0)$, $\hat{\rho}_S^{11}(0) = |c_1|^2 \rho_S(0)$ and the posterior intensity

$$\begin{aligned}
K_t = & \text{Tr}_{\mathfrak{h}_S} ((L + L^\dagger)\hat{\rho}(t)) \\
& + \text{Tr}_{\mathfrak{h}_S} (S(\alpha_0(t)\hat{\rho}_S^{00}(t) + \alpha_1(t)\hat{\rho}_S^{11}(t))) \\
& + \text{Tr}_{\mathfrak{h}_S} (S^\dagger(\alpha_0^*(t)\hat{\rho}_S^{00}(t) + \alpha_1^*(t)\hat{\rho}_S^{11}(t))).
\end{aligned}$$

E. Numerical Example

Let us consider the coherent Gaussian pulses:

$$\begin{aligned}
\alpha_0(t) = & \left(\frac{2\Omega^2}{\pi} \right)^{1/4} \exp \left[-\frac{\Omega^2}{4} (t-3)^2 \right], \\
\alpha_1(t) = & \left(\frac{2\Omega^2}{\pi} \right)^{1/4} \exp \left[-\frac{\Omega^2}{4} (t-5)^2 \right].
\end{aligned}$$

with the parameters $\Omega = 2.4\kappa$ and $\kappa = 1$. As the system interacting with the Bose field we take a two level atom and assume that $L = \sqrt{\kappa}\sigma_-$, $S = I$, and $H_S = 0$. The values of parameters guaranties us the most efficient excitation [8]. In Fig. 2 the probability of being in the excited state for the two level atom as well as the probability of at least one count in the interval $(0, t]$ are shown for two chosed of initial conditions. One can check that in the limit we have

$$\begin{aligned}
\lim_{t \rightarrow +\infty} P_0^t(0) = & \langle 0|\rho(0)|0 \rangle \left[p e^{-\int_0^{+\infty} |\alpha_0(t)|^2 dt} \right. \\
& \left. + (1-p) e^{-\int_0^{+\infty} |\alpha_1(t)|^2 dt} \right].
\end{aligned}$$

IV. CONCLUSIONS

We have derived the filtering equations for the two kinds of non-classical states of the Bose field, namely for a combination of vacuum and single photon states and for a mixture of two coherent states. To generate the Bose field in the desired non-classical states we have used ancilla systems. It should be stressed that the models of signal generators are only theoretical tools which serve for derivation of the filtering equations for the non-classical signals. Our filtering equation for the input field in a single photon state agrees with the filtering equation derived in [26], but for the Bose field taken in a combination of vacuum and single photon states our results are not consistent with [26]. We have got the same master equations but different filters. Unlike us, authors of [26] to solve the problem put their two-level ancilla system initially in the excited state. Consequence of different choice of the initial state of ancilla are different intensities of output processes determining the filter and finally different trajectories.

We note that our calculations for a mixture of two coherent states can be easily generalized to the case of a

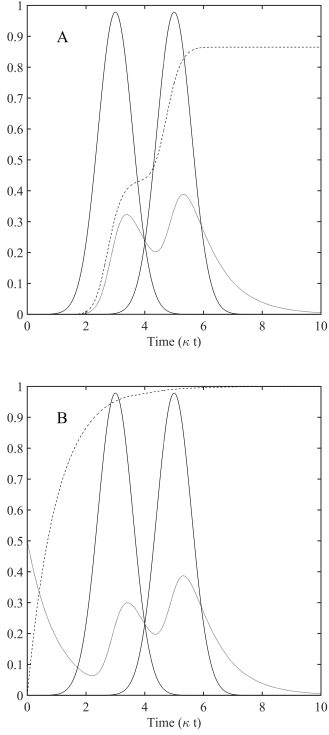


FIG. 2: The probability of being in the excited state for the two level atom. The solid lines are $|\alpha_1(t)|^2/2$ and $|\alpha_2(t)|^2/2$, the dashed line is the excitation calculated from the master equation, and the dotted line is the probability of at least one count in the interval $(0, t]$ for the Bose field taken in a mixture of two coherent states for $p = 0.5$, and for the two-level system being initially in the ground state (A) and in the upper state (B).

V. ACKNOWLEDGEMENTS

This paper was partially supported by the National Science Center project 2015/17/B/ST2/02026.

-
- [1] R. Loudon, *The Quantum Theory of Light*, third edition, (Oxford University Press, Oxford, 2000).
 - [2] V. Giovannetti, S. Lloyd, and L. Maccone, *Nat. Photon.* **4**, 222 (2011).
 - [3] I. D. Leroux, M. H. Schleier-Smith, H. Zhang, and V. Vuletić, *Phys. Rev. A* **85**, 013803 (2012).
 - [4] E. Knill, R. Laflamme, and G. J. Milburn, *Nature* **409**, 46 (2001).
 - [5] P. P. Rohde, T. C. Ralph, and M. A. Nielsen, *Phys. Rev. A* **72**, 052332 (2005).
 - [6] M. Stobińska, G. Alber, and G. Leuchs, *EPL* **86**, 14007 (2009).
 - [7] M. Stobińska, G. Alber, and G. Leuchs, *Advances in Quantum Chemistry*, Vol. **60**, pp. 201-226, (2010).
 - [8] Y. Wang, J. Minář, L. Sheridan, and V. Scarani, *Phys. Rev. A* **83**, 063842 (2011).
 - [9] P. V. Elyutin, *Phys. Rev. A* **85** 033816 (2012).
 - [10] I. M. Mirza and J. C. Schotland, *Phys. Rev. A* **94**, 012309 (2016).
 - [11] B. Q. Baragiola, R. L. Cook, A. M. Brańczyk, and J. Combes, *Phys. Rev. A*, **86**, 013811 (2012).
 - [12] K. M. Gheri, K. Ellinger, T. Pellizzari, and P. Zoller, *Fortschr. Phys.* **46** 4-5, 401-415, (1998).
 - [13] V. P. Belavkin, Vol. **378** of the series *Lecture Notes in Physics*, pp. 151-163, 2005
 - [14] R. L. Hudson and K.R. Parthasarathy, *Commun. Math. Phys.* **93**, 301 (1984).
 - [15] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, (Basel: Birkhäuser Verlag, 1992).
 - [16] A. Barchielli, in *Lecture Notes Math.* 1882, pp. 207-291, (Springer, Berlin, 2006)
 - [17] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, (Oxford University Press, New York, 2002).
 - [18] C.W. Gardiner and P. Zoller, *Quantum noise* (Springer-Verlag Berlin-Heidelberg, 2010).
 - [19] H. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer-Verlag Berlin-Heidelberg, 1993).
 - [20] H. M. Wiseman and G. J. Milburn, *Phys. Rev. A* **47**, 642-662 (1993).
 - [21] H. M. Wiseman and G. J. Milburn, *Quantum measurement and control*, (Cambridge University Press, 2010).
 - [22] A. Barchielli and V.P. Belavkin, *J. Phys. A: Math. Gen.* **24**, 1495 (1991).

- [23] L. Bouten, M. Guță, H. Maassen, J. Phys. A: Math. Gen. **37** 3189 (2004).
- [24] A. Dąbrowska and J. E. Gough, Russian J. Math. Phys. **23**, 172 (2016).
- [25] J. E. Gough, M. R. James, H. I. Nurdin, 2011 50th IEEE Conference on Decision and Control and European Control Conference, 5570 - 5576, (2011).
- [26] J. E. Gough, M. R. James, H. I. Nurdin, J. Combes, Phys. Rev. A **86**, 043819 (2012).
- [27] G. J. Milburn, Eur. Phys. J. Spec. Top. **159**, 113 (2008).
- [28] G. J. Milburn, Springer Handbook of Laser and Optics, Träger (Ed.), 2nd edition, Chap. 18.2, pp. 1307-1310, Springer, 2012
- [29] C. W. Gardiner and M. J. Collet, Phys. Rev. A **31**, 3761 (1985).